STEADY CONVECTION IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

(STATSIONARNAIA KONVEKTSIIA PRI NALICHII VNESHNEGO MAGNITNOGO POLIA)

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The problem of the equilibrium of a liquid enclosed in a vessel heated from below has been considered by Sorokin [1], Iudovich and Ukhovskii [2] and Velt [3]. It has been established that if the Rayleigh number λ exceeds a certain critical value λ_0, then secondary steady flows arise in the liquid.

The stability of a conductive liquid heated from below has been studied by many authors. The most complete and general studies are those of Sorokin and Sushkin [4], whose paper contains the appropriate bibliography, and that of Shliomis [5]. The results of [4 and 5] make clear the physical picture of the phenomena associated with the heating of a conductive fluid and indicate the possible existence of secondary steady and periodic flows.

The existence of steady convective flows in a conductive liquid are proved below. Our study is based on the procedure set forth in [2].

1. Let us assume that the density ρ^* of the liquid is related to the temperature T^* in linear fashion

$$\rho^* = \rho_0 (1 - \alpha \Delta T^*), \qquad \Delta T^* = T^* - T_0^*$$

Here α is the coefficient of volume expansion and ρ_0 is the density of the liquid at the temperature T_0^*.

We know that a heated liquid can be in equilibrium only if the temperature T^* at the point $\mathbf{r}^*$ is of the form $T^* = T_0^* + \beta \mathbf{l r}^*$, where $\mathbf{l}$ is a unit vector in the direction opposite that of the gravitational force.

Henceforth we shall assume β to be positive, writing the local temperature in the liquid in the form

$$T^* = T_0^* + \beta \mathbf{l r}^* + \theta^*$$

We shall consider the steady motions of the liquid in the bounded region Ω for a constant temperature gradient and a constant magnetic field in the external medium. We introduce the dimensionless variables

$$\mathbf{u}^* = \frac{\nu}{L}\mathbf{u}, \qquad \mathbf{h}^* = \left(\frac{\rho_0 \nu^3}{\mu_e L^2 \eta}\right)^{1/2}\mathbf{h}, \qquad T^* = \frac{\beta L \nu}{\varkappa} T, \qquad \theta^* = \frac{\beta L \nu}{\varkappa}\theta, \qquad x_i^* = L x_i$$

$$\left(\varkappa = \frac{k \rho_0}{c_v}, \quad \eta = (\mu_e \sigma)^{-1}\right)$$

Here ν is the kinematic viscosity; L is the characteristic linear dimension; $\mathbf{u}$, $\mathbf{h}$ are, respectively, the vectors of the liquid's velocity and of the intensity of the magnetic field induced by the liquid's motion; x_i are Cartesian coordinates; k is the coefficient of heat conduction and c_v the specific heat; η is the coefficient of magnetic viscosity; σ is the electric conductivity; μ_e is the magnetic permeability. The asterisk denotes dimensional quantities. The equations of the steady motion of the liquid [6] become

$$\triangle \mathbf{u} = -\lambda\theta\mathbf{l} - R_m \mathbf{h}_{x_3} + \nabla\Phi + (\mathbf{u}\cdot\nabla)\mathbf{u} - R_\sigma(\mathbf{h}\cdot\nabla)\mathbf{h}, \qquad \triangle\theta = -\mathbf{u}_l + P(\mathbf{u}\cdot\nabla)\theta$$

$$\triangle\mathbf{h} = -R_m\mathbf{u}_{x_3} + R_\sigma(\mathbf{u}\cdot\nabla)\mathbf{h} - R_\sigma(\mathbf{h}\cdot\nabla)\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0$$

$$\Phi = \frac{L^2}{\nu^2}\left[\frac{p^*}{\rho_0} + \frac{\mu_e H}{2\rho_0} - g\mathbf{l}\mathbf{r}^* + \frac{1}{2}\,\alpha\beta g(\mathbf{l}\mathbf{r}^*)^2\right] \tag{1.1}$$

$$\lambda = \frac{\alpha\beta g L^4}{\varkappa\nu}, \qquad P = \frac{\nu}{\varkappa}, \qquad R_m = HL\left(\frac{\mu_e}{\rho_0\nu\eta}\right)^{1/2}, \qquad \mathbf{u}_{x_3} \equiv \frac{\partial\mathbf{u}}{\partial x_3}, \qquad u_l = \mathbf{u}\cdot\mathbf{l}$$

Here λ is the Rayleigh number, P is the Prandtl number, R_σ is the magnetic Reynolds number, R_m is the magnetic pressure number, p is the pressure, g is the acceleration due to gravity, H is the magnitude of the external magnetic field intensity, and $-u_l$ is the projection of the velocity on the direction of gravity. The x_3-axis is directed along the external magnetic field; in addition, the usual convention as regards the omission of the sign indicating summation over a recurrent index is observed.

The first equation of (1.1) yields the dynamic equations, the second the heat conduction equations, the third the induction equations, and the fourth the incompressibility equations.

If the vessel is completely filled and the vessel wall is an ideal conductor, then the boundary conditions with a constant temperature gradient in the external medium are as follows:

$$\theta = 0, \quad \mathbf{u} = 0, \quad \mathbf{h}\cdot\mathbf{n} = 0, \quad \operatorname{rot}\mathbf{h}\cdot\boldsymbol{\tau} = 0 \tag{1.2}$$

Here $\mathbf{n}$ is the normal and $\boldsymbol{\tau}$ is an arbitrary vector tangent to the vessel wall $\partial\Omega$.

Equations (1.1) under boundary conditions (1.2) have a trivial solution corresponding to the liquid at rest,

$$\mathbf{u} = \mathbf{h} = 0, \quad \theta = 0 \qquad (T^* = \dot{T}_0^* + \beta\mathbf{l}\mathbf{r}^*) \tag{1.3}$$

Along with problem (1.1),(1.2) we shall consider the corresponding linearized steady-state problem

$$\Delta\mathbf{u} = -\lambda\theta\mathbf{l} - R_m\mathbf{h}_{x_3} + \nabla\Phi, \;\; \Delta\mathbf{h} = -R_m\mathbf{u}_{x_3}, \;\; \Delta\theta = -u_l, \;\; \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0 \tag{1.4}$$

in Ω with boundary condition (1.2) at $\partial\Omega$.

2. Let us now define some function spaces. By H_u we denote the Hilbert space which is the closure of the set of sufficiently smooth solenoidal vectors finite in Ω in the norm generated by the scalar product

$$(\mathbf{u}, \mathbf{v})_{H_u} = \int_\Omega \mathbf{u}_{x_i}\mathbf{v}_{x_i}\,dx = \int_\Omega \operatorname{rot}\mathbf{u}\cdot\operatorname{rot}\mathbf{v}\,dx \tag{2.1}$$

The pace H_h is the subspace which is a closure in the norm generated by the scalar product (2.1) of the set of continuously differentiable solenoidal vectors for which $\mathbf{h}\cdot\mathbf{n} = 0$ on $\partial\Omega$.

The space H_θ is the closure of the set of sufficiently smooth functions finite in Ω in the norm generated by the scalar product

$$(\theta, \Phi)_{H_\theta} = \int_\Omega \nabla\theta\cdot\nabla\Phi\,dx \tag{2.2}$$

We know [7 and 8] that H_u, H_h, H_θ are imbeddable in L_2. Thus, the above norms in the corresponding spaces are equivalent to the conventional norm W_2^1.

Let us also introduce the Hilbert space H whose elements $\mathbf{f}$ are pairs $\mathbf{u} \in H_u$, $\mathbf{h} \in H_h$, and where the scalar products $\mathbf{f} = \{\mathbf{u},\mathbf{h}\}$ and $\varphi = \{\mathbf{v},\psi\}$ are defined by Formula

$$(\mathbf{f}, \varphi)_H = (\mathbf{u}, \mathbf{v})_{H_u} + (\mathbf{h}, \psi)_{H_h} \tag{2.3}$$

The generalized solution of problem (1.1),(1.2) is the triplet $\mathbf{u} \in H_u$, $\mathbf{h} \in H_h$, $\theta \in H_\theta$, which satisfies the integral identities

$$(\mathbf{u}, \mathbf{v})_{H_u} = \lambda \int_\Omega \theta u_l \, dx + R_m \int_\Omega \mathbf{h}_{x_3} \mathbf{v} \, dx - \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u}\mathbf{v} \, dx + R_\sigma \int_\Omega (\mathbf{h} \cdot \nabla) \mathbf{h}\mathbf{v} \, dx$$

$$(\mathbf{h}, \psi)_{H_h} = R_m \int_\Omega \mathbf{u}_{x_3} \psi \, dx - R_\sigma \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{h}\psi \, dx + R_\sigma \int_\Omega (\mathbf{h} \cdot \nabla) \mathbf{u}\psi \, dx \tag{2.4}$$

$$(\theta, \Phi)_{H_\theta} = \int_\Omega u_l \Phi \, dx - P \int_\Omega (u \cdot \nabla) \theta \Phi \, dx, \quad \mathbf{v} \in H_u, \quad \psi \in H_h, \quad \Phi \in H_\theta$$

The results of Ladyzhenskaia and Solonnikov [7 and 9] indicate that the generalized solutions of problem (1.1),(1.2) are doubly continuously differentiable (*) in Ω and that they satisfy boundary conditions (1.2).

The generalized solution for linear problem (1.4) is determined in the same way.

3. Let us reduce problem (2.4) to operator equations. For sufficiently smooth functions $\mathbf{F}_u$, $\mathbf{F}_h$, θ, $\mathbf{F}$ we define the operators K_u, K_h, K_θ, K_f by the requirement that the integral identities

$$(K_u \mathbf{F}_u, \mathbf{v})_{H_u} = \int_\Omega \mathbf{F}_u \mathbf{v} \, dx, \quad (K_h \mathbf{F}_h, \psi)_{H_h} = \int_\Omega \mathbf{F}_h \psi \, dx, \quad (K_\theta \theta, \Phi)_{H_\theta} = \int_\Omega \theta \Phi \, dx \tag{3.1}$$

$$(K_f \mathbf{F}, \mathbf{f})_H = (K_u \mathbf{F}_u, \mathbf{v})_{H_u} + (K_h \mathbf{F}_h, \psi)_{H_h} \tag{3.2}$$

be fulfilled for any $\mathbf{v} \in H_u$, $\psi \in H_h$, $\Phi \in H_\theta$, $\mathbf{f} \in H$.

L e m m a 3.1 . The operators K_f, K_u, K_h, K_θ are bounded and completely continuous. Let us prove this for the operator K_u . The boundedness of the operator follows from the estimate

$$|(K_u \mathbf{F}, \mathbf{v})_{H_u}| = \left| \int_\Omega \mathbf{F}\mathbf{v} \, dx \right| \leqslant \left(\int_\Omega \mathbf{F}^2 \, dx \right)^{1/2} \left(\int_\Omega \mathbf{v}^2 dx \right)^{1/2} \tag{3.3}$$

and from the imbedding of H_u in L_2 . The complete continuity [8 and 10] of the imbedding from H_u in L_2 and estimate (3.3) imply the complete continuity of K_u .

The heat conduction equation becomes the operator equation

$$\theta + PK_\theta (\mathbf{u} \cdot \nabla) \theta = K_\theta u_l \tag{3.4}$$

With $u_l = 0$ the homogeneous equation $\theta + PK_\theta (\mathbf{u} \cdot \nabla) \theta = 0$ has only a trivial solution in H_θ . In fact, taking its scalar product with θ , we have

$$(\theta, \theta)_{H_\theta} + P (K_\theta (\mathbf{u} \cdot \nabla) \theta, \theta)_{H_\theta} = (\theta, \theta)_{H_\theta} + P \int_\Omega (\mathbf{u} \cdot \nabla) \theta\theta \, dx =$$

$$= (\theta, \theta)_{H_\theta} - P \int_\Omega {}^1/_2 \theta^2 \nabla \cdot \mathbf{u} \, dx = (\theta, \theta)_{H_\theta} = 0$$

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*) Following Vorovich and Iudovich [8], we can prove that derivatives of the functions $\mathbf{u}$, $\mathbf{h}$, θ of any order are continuous in the closed region Ω if the boundary $\partial\Omega$ is sufficiently smooth.

The Fredholm theorem implies that (3.4) is solvable for any $\mathbf{u} \in H_u$. Equation (3.4) is determined by the operator

$$\theta = A\mathbf{u}.$$

We shall now show that A is a bounded operator acting from H_u into H_θ.

Taking the scalar product of (3.4) and θ, we find, as above, that

$$(\theta, \theta)_{H_\theta} = \int_\Omega u_l \theta \, dx \leqslant \|\mathbf{u}\|_{L_2} \|\theta\|_{L_2}$$

The imbedding of H_θ in L_2 implies the boundedness of the operator A.

System (2.4) becomes the system

$$\begin{aligned} \mathbf{u} - R_m K_u \mathbf{h}_{x_3} &= \lambda K_u \theta \mathbf{u} - K_u (\mathbf{u}\cdot\nabla)\mathbf{u} + R_\sigma K_u (\mathbf{h}\cdot\nabla)\mathbf{h} \\ \mathbf{h} - R_m K_h \mathbf{u}_{x_3} &= -R_\sigma K_h (\mathbf{u}\cdot\nabla)\mathbf{h} + R_\sigma K_h (\mathbf{h}\cdot\nabla)\mathbf{u} \end{aligned} \tag{3.5}$$

or, in the space H, Equation

$$\mathbf{f} - R_m K_1 \mathbf{f} = \lambda K_2 \mathbf{f} + K_3 \mathbf{f} \qquad (K_2 \mathbf{f} \equiv K_u A \mathbf{u})$$

L e m m a 3.2. The operators K_1, K_2, K_3 are bounded and completely continuous.

The Lemma follows from the complete continuity of the operators K_u and K_h. Let us show, for example, the complete continuity of the operator $Bf \equiv K_h (\mathbf{u}\cdot\nabla)\mathbf{h}$, which acts from H into H_h. We have

$$(B\mathbf{f}, \psi)_{H_h} \equiv (K_h (\mathbf{u}\cdot\nabla)\mathbf{h}, \psi)_{H_h} = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{h}\psi \, dx = -\int_\Omega \mathbf{h} (\mathbf{u}\cdot\nabla)\psi \, dx \tag{3.6}$$

From this we have the estimate

$$(B\mathbf{f}, \psi)_{H_h} \leqslant C_1 \|\psi\|_{H_h} \|\mathbf{u}\|_{L_4} \|\mathbf{h}\|_{L_4}$$

Replacing φ by $B\mathbf{f}$, we obtain

$$\|B\mathbf{f}\|_H \leqslant C \|\mathbf{u}\|_{H_u} \|\mathbf{h}\|_{H_h} \leqslant C \|\mathbf{f}\|_H^2 \tag{3.7}$$

which implies the boundedness of the operator B.

From (3.6) we find (*) that for some sequence $\mathbf{f}^{(n)}$

$$|(B\mathbf{f}^{(n)} - B\mathbf{f}^{(m)}, \psi)_{H_h}| \leqslant C (\|\mathbf{f}^{(n)}\|_{L_4} + \|\mathbf{f}^{(m)}\|_{L_4}) \|\psi\|_{H_h} \|\mathbf{f}^{(n)} - \mathbf{f}^{(m)}\|_{L_4}$$

When ψ has been replaced by $B\mathbf{f}^{(n)} - B\mathbf{f}^{(m)}$ the complete continuity of the operators [8 and 10] from H_h into L_4 implies the complete continuity of the operator B.

The operator in the left-hand side of (3.5) is invertible.

In fact, taking the scalar product of (3.5) and $\mathbf{f}$ and setting the right-hand side equal to zero, we obtain

$$(\mathbf{f}, \mathbf{f})_H - R_m (K_1 \mathbf{f}, \mathbf{f})_H = 0$$

but

$$(K_1 \mathbf{f}, \mathbf{f})_H \equiv (K_u \mathbf{h}_{x_3}, \mathbf{u})_{H_u} + (K_h \mathbf{u}_{x_3}, \mathbf{h})_{H_h} = \int_\Omega (\mathbf{h}_{x_3}\mathbf{u} + \mathbf{u}_{x_3}\mathbf{h}) \, dx = 0 \tag{3.8}$$

So that

$$(\mathbf{f}, \mathbf{f})_H = 0, \qquad \mathbf{f} \equiv 0$$

By the Fredholm theorem, the completely continuous operator $I - R_m K_1$ has an inverse L which is bounded. In fact, by virtue of (3.8),

$$\|\mathbf{f} - R_m K_1 \mathbf{f}\|_H = (\|\mathbf{f}\|_H^2 + R_m^2 \|K_u \mathbf{h}_{x_3}\|_{H_u}^2 + R_m^2 \|K_h \mathbf{u}_{x_3}\|_{H_h}^2)^{1/2} \geqslant \|\mathbf{f}\|_H$$

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*) Notation: $\|\mathbf{f}\|_{L_4} = \|\mathbf{u}\|_{L_4} + \|\mathbf{h}\|_{L_4}$.

This implies [11] the boundedness and (by virtue of its linearity) the continuity of the operator L .

System (3.5) or (1.1) is thus equivalent to the operator equation

$$\mathbf{f} = \lambda L K_2 \mathbf{f} + L K_3 \mathbf{f} \equiv K\ (\mathbf{f}, \lambda) \tag{3.9}$$

Similarly, linear system (1.4) becomes the system

$$\mathbf{u} - R_m K_u \mathbf{h}_{x_3} = \lambda K_u (K_\theta u_l)\,\mathbf{l}, \qquad \mathbf{h} - R_m K_h \mathbf{u}_{x_3} = 0 \tag{3.10}$$

with the corresponding operator equation

$$\mathbf{f} = \lambda L K_u (K_\theta u_l)\,\mathbf{l} \tag{3.11}$$

Since the operator L is continuous and the operators K_2 and K_3 completely continuous, the operator K in (3.9) is completely continuous. We shall show that the right-hand side of (3.11) is the Frechet differential of the operator K . To do this we must demonstrate that

$$\| K(\mathbf{f}, \lambda) - \lambda L K_u (K_\theta u_l)\,\mathbf{l} \|_H = \| \lambda L K_2 \mathbf{f} - \lambda L K_u (K_\theta u_l) \mathbf{l} + L K_3 \mathbf{f} \|_H \leqslant C \| \mathbf{f} \|_H^2$$

By virtue of the linearity of the operators L and K_u it is sufficient to estimate

$$\| A\mathbf{u} - K_\theta u_l \|_{H_\theta}, \qquad \| K_3 \mathbf{f} \|_H$$

The estimate of the operator K_3 follows from (3.7), while for the difference $A\mathbf{u} - K_\theta u_l$ (3.4) gives us

$$|(A\mathbf{u} - K_\theta u_l,\ \Phi)_{H_\theta}| = P \left| \int_\Omega (\mathbf{u}\cdot\nabla)\,\theta\Phi\, dx \right| \leqslant C_1 \|\theta\|_{H_\theta} \|\mathbf{u}\|_{H_u} \|\Phi\|_{H_\theta}$$

Setting $\Phi = A\mathbf{u} - K_\theta u_l$ and making the substitution $\theta = A\mathbf{u}$, by virtue of the boundedness of the operator A we have

$$\| A\mathbf{u} - K_\theta u_l \|_{H_\theta} \leqslant C \|\mathbf{u}\|_{H_u}^2 \leqslant C \|\mathbf{f}\|_H^2$$

4. Let us consider the possibility of the existence of steady-state solutions (1.1),(1.2) which are different from (1.4). Let

$$\lambda_0 = \inf \frac{(\mathbf{u}, \mathbf{u})_{H_u}}{(K_\theta u_l,\ K_\theta u_l)_{H_\theta}} \tag{4.1}$$

where the lower bound is taken over all the solenoidal vector functions $\mathbf{u} \in H_u$.

In [2 and 3] it is shown that λ_0 is the critical Rayleigh number for the steady-convection equations. If an external magnetic field is present, the following theorem is valid.

Theorem 4.1. If problem (1.1),(1.2) has a nontrivial solution, then $\lambda > \lambda_0$.

Let problem (1.1),(1.2) have a nontrivial solution. Taking the scalar products of (1.1) and $\mathbf{u}$, $\mathbf{h}$, λ, θ and adding, we obtain

$$(\mathbf{u}, \mathbf{u})_{H_u} + (\mathbf{h}, \mathbf{h})_{H_h} + \lambda\, [(\theta, \theta)_{H_\theta} - 2\,(u_l, \theta)] = 0 \tag{4.2}$$

As is evident from (4.2) and the unique solvability of (3.4), the solution of problem (1.1),(1.2) differs from zero only for $u_l \neq 0$.

It is well known [12] that

$$\min\, [(\theta, \theta)_{H_\theta} - 2\,(u_l, \theta)] = -(K_\theta u_l,\ K_\theta u_l)_{H_\theta}$$

where the minimum is taken over all $\theta \in H_\theta$.

From (4.2) it follows that:

$$(\mathbf{u}, \mathbf{u})_{H_u} + (\mathbf{h}, \mathbf{h})_{H_h} - \lambda\,(K_\theta u_l,\ K_\theta u_l)_{H_\theta} \leqslant 0$$

and by virtue of (4.1) we find that $\lambda_0 - \lambda < 0$. The theorem has been proved.

The theorem implies that with $\lambda \leq \lambda_0$, problem (1.1),(1.2) has only a trivial solution. Thus, the critical Rayleigh number does not diminish upon the imposition of an external magnetic field. The constant magnetic field stabilizes the equilibrium of the liquid.

Let us make use of the theory of bifurcations of nonlinear operator equations [13] in our search for steady-state solutions of (3.9) which differ from (1.3).

The real number λ_1 is called the bifurcation point of the operator K if for any ε, $\delta > 0$ it is possible to indicate an eigenvalue λ of the operator K such that $|\lambda - \lambda_1| < \delta$ and that Equation (3.9) has at least one eigenvector $\mathbf{f}$ such that $\|\mathbf{f}\|_H < \varepsilon$.

The results of Krasnosel'skii [13] imply that the bifurcation points of the operator K can only be the eigenvalues of its Frechet differential (3.10).

If λ_1, an eigenvalue of problem (3.10), is of odd multiplicity (*), then λ_1 is the bifurcation point of the operator K. Corresponding to this point is a continuous branch of the eigenvectors of operator K. The parameter λ is real and positive.

5. Let us prove the existence of positive eigenvalues of (3.11). Operator equation (3.11) is equivalent to system (3.10).

Replacing $\mathbf{u}$ in the dynamic equation by its value as determined from the induction equation, we reduce (3.10) to an operator equation for $\mathbf{u}$,

$$\mathbf{u} - R_m{}^2 K_u \frac{\partial}{\partial x_3} K_h \frac{\partial \mathbf{u}}{\partial x_3} = \lambda K_u (K_\theta u_l)\, \mathbf{l} \tag{5.1}$$

After determining $\mathbf{u}$ from (5.1), we find $\mathbf{h}$ and θ from the induction and heat conduction equations,

$$\mathbf{h} = R_m K_h \mathbf{u}_{x_3}, \qquad \theta = K_\theta u_l$$

The operator in the right- and left-hand sides of (5.1) are linear, positive, completely continuous, and selfadjoint in H_u. In fact,

$$-\left(K_u \frac{\partial}{\partial x_3} K_h \frac{\partial \mathbf{u}}{\partial x_3}, \mathbf{v}\right)_{H_u} = -\int_\Omega \frac{\partial}{\partial x_3} K_h \frac{\partial \mathbf{u}}{\partial x_3} \mathbf{v}\, dx = \int_\Omega K_h \frac{\partial \mathbf{u}}{\partial x_3} \frac{\partial \mathbf{v}}{\partial x_3}\, dx =$$

$$= \left(K_h \frac{\partial \mathbf{u}}{\partial x_3},\ K_h \frac{\partial \mathbf{v}}{\partial x_3}\right)_{H_h}$$

$$(K_u (K_\theta u_l)\, \mathbf{l}, \mathbf{v})_{H_u} = \int_\Omega K_\theta u_l v_l\, dx = (K_\theta u_l,\ K_\theta v_l)_{H_\theta}$$

This implies the following theorems and lemma.

T h e o r e m 5.1 . There exists a denumerable number of eigenvalues of system (1.4) $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \to +\infty$. The corresponding system of functions $(\mathbf{u}_n, \mathbf{h}_n)$ is complete in H. (The system θ_n is complete in H_0).

L e m m a 5.1 . The eigenfunctions which corresond to various eigenvalues λ and λ^* of Equations (1.4) satisfy the following orthogonality conditions:

$$(\mathbf{u}, \mathbf{u}^*)_{H_u} + (\mathbf{h}, \mathbf{h}^*)_{H_h} = (\mathbf{f}, \mathbf{f}^*)_H = 0, \qquad (\theta, \theta^*)_{H_\theta} = 0 \tag{5.2}$$

Any eigenvalue λ_k with an odd number of associated eigenvectors is a bifurcation point of Equation (1.1).

As in [2 and 12], the problem of finding the eigenvalues and eigenfunctions of (5.1) can be reduced to the problem of minimizing the functional(**)

*) The multiplicity of the eigenvalue λ of the operator K is the dimensionality of the subspace spanning the eigen- and adjoint vectors corresponding to the characteristic number λ.

**) The equivalence of problem (1.4) to a variational problem (which differs somewhat from (5.3)) is demonstrated in [4].

$$\lambda = J(\mathbf{u}) = \frac{(\mathbf{u}, \mathbf{u})_{H_u} + R_H{}^2 (K_h \mathbf{u}_{x_s}, K_h \mathbf{u}_{x_s})}{(K_\theta u_l, K_\theta u_l)} \qquad (u \in H_u) \tag{5.3}$$

Theorem 5.2. The problem of finding the eigenvalues and eigenfunctions of system (1.4) is equivalent to the minimum problem for functional (5.3).

Let us show that functional (5.2) is bounded from below.

In fact, by the Cauchy-Buniakowski inequality,

$$(K_\theta u_l, K_\theta u_l)_{H_\theta}^2 \leqslant \Big(\int_\Omega K_\theta u_l u_l \, dx\Big)^2 \leqslant \Big(\int_\Omega u_l{}^2 \, dx\Big)\Big(\int_\Omega \theta^2 \, dx\Big)$$

and making use of the Poincaré inequality,

$$\int_\Omega \theta^2 \, dx \leqslant C_1 \int_\Omega \nabla\theta \cdot \nabla\theta \, dx = C_1 (K_\theta u_l, K_\theta u_l)_{H_\theta}$$

and the theorem of imbedding of $\mathring{H}_u$ in L_2 we obtain

$$(K_\theta u_l, K_\theta u_l) \leqslant C (\mathbf{u}, \mathbf{u})_{H_u} \tag{5.4}$$

From (5.3) and (5.4) we find that

$$J(\mathbf{u}) \geqslant C^{-1}$$

The minimum problem for functional (5.3) has as its consequence the following theorem [12].

Theorem 5.3. Let λ_1 be the exact lower bound of the functional $J(\mathbf{u})$. Then, there exists a vector-function $\mathbf{u}_1 \in H_u$ such that $J(\mathbf{u}_1) = \lambda_1$, where λ_1 is the smallest eigenvalue, and $\mathbf{u}_1$ ($\mathbf{h}_1$, θ_1, respectively) is the eigenfunction of system (1.4).

Theorem 5.4. Let $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ be the eigenvalues of (1.4), and let (u_s, h_s) be their associated eigenfunctions orthonormalized in the sense of (5.2). Then, there exists a function $\mathbf{u}_{n+1} \in H_u$, which minimizes functional (5.2) under the additional conditions

$$(\mathbf{u}_{n+1}, \mathbf{u}_m)_{H_u} + (\mathbf{h}_{n+1}, \mathbf{h}_m)_{H_h} = 0, \qquad (\theta_{n+1}, \theta_m)_{H_\theta} = 0 \qquad (m = 1, 2 \ldots n)$$

where $\mathbf{h}_{n+1}$, θ_{n+1} can be determined from $\mathbf{u}_{n+1}$ from the induction and heat conduction equations.

The triplet $\mathbf{u}_{n+1}$, $\mathbf{h}_{n+1}$, θ_{n+1} is the eigenfunction of (1.4) which corresponds to the number

$$\lambda_{n+1} = J(\mathbf{u}_{n+1})$$

In actual computations, it is more convenient to write (5.2) in the form

$$\lambda = J(\mathbf{u}) = \Big(\int_\Omega \mathbf{u}_{x_i} \mathbf{u}_{x_i} \, dx + \int_\Omega \mathbf{h}_{x_i} \mathbf{h}_{x_i} \, dx\Big)\Big(\int_\Omega \theta_{x_i} \theta_{x_i} \, dx\Big)^{-1}$$

Here $\mathbf{h}$ and θ are the solutions of the linear induction and heat conduction equations of (1.4) with boundary conditions (1.2). Similar results are obtainable in the case where the liquid is enclosed in a dielectric.

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REMARK ON THE PAPERS BY R.V.BIRIKH

"ON THE SPECTRUM OF SMALL PERTURBATIONS OF PLANE-PARALLEL COUETTE FLOW" PMM Vol.29, № 4, 1965, and "ON SMALL PERTURBATIONS OF A PLANE-PARALLEL FLOW WITH CUBIC VELOCITY PROFILE" PMM Vol.30, № 2, 1966

(ZAMECHANIE K RABOTAM R.V.BIRIKHA

"O spektre malykh vozmushchenii ploskoparallel'nogo techeniia Kuetta"

PMM T.29, Vyp.4, 1965, i

"O malykh vozmushcheniiakh ploskoparallel'nogo techeniia s kubicheskim profilem skorosti" PMM T.30, Vyp.2, 1966)

PMM Vol.30, № 6, 1966, p.1147

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In the second of the above papers, when the spectrum of decrements of normal perturbations of a flow with cubic velocity profile was discussed, the possibility was indicated of the existence of a vibrational instability in this flow at high Reynolds numbers. In order to verify this hypothesis, a new